Direct Sum

define Let $E, F : \mathbb{V}ec_{\mathbb{F}}$.

We define the direct sum $E \oplus F$.

Let the underlying set of $E \oplus F$ be

$$E \times F.$$

Define vec. space structure:

$$+ : E \oplus F \times E \oplus F \longrightarrow E \oplus F$$
$$\langle\langle x, y\rangle, \langle x', y'\rangle\rangle \rightsquigarrow \langle x + x', y + y'\rangle$$

$$\cdot : \mathbb{F} \times E \oplus F \longrightarrow E \oplus F$$
$$\langle k, \langle x, y\rangle\rangle \rightsquigarrow \langle k\,x, k\,y\rangle$$

Then $E \oplus F : \mathbb{V}ec_{\mathbb{F}}$.

define We have a few natural maps associated with $E \oplus F$.

Projections

$$E \xleftarrow{fst} E \oplus F \xrightarrow{snd} F$$
$$x \longleftarrow \langle x, y\rangle \longrightarrow y$$

Injections

$$E \xrightarrow{injl} E \oplus F \xleftarrow{injr} F.$$
$$x \rightsquigarrow \langle x, 0\rangle \qquad \langle 0, y\rangle \longleftarrow y$$

define Let $S_1, S_2 \subseteq E$ be subspaces.

Then

$$S_1 + S_2 \subseteq E$$

is the smallest subspace containing both S_1 and S_2.

Let $S_1 \cap S_2 = \{0\}$, $S_1 + S_2 = E$.

Let us consider S_1 and S_2 as vector spaces.

Then

$$S_1 \oplus S_2 \xrightarrow{i} E$$
$$\langle x, x'\rangle \rightsquigarrow x + x'$$

Then we have

$$S_1 \xleftarrow{fst} S_1 \oplus S_2 \xrightarrow{snd} S_2$$

for all $x : S_1, x' : S_2$.

$$x \longleftarrow x + x' \longrightarrow x'$$

$$S_1 \xrightarrow{inl} S_1 \oplus S_2 \xleftarrow{inr} S_2$$
$$x \rightsquigarrow x \qquad x' \longleftarrow x'$$

Product

define Let $E \xleftarrow{f} G \xrightarrow{g} F$.

We define

$$G \xrightarrow{\frac{f}{g}} E \oplus F$$
$$z \rightsquigarrow \langle f\,z, g\,z\rangle$$

We'll often write

$$z\,\frac{f}{g} := \begin{pmatrix} f \\ g \end{pmatrix} z,$$

So

$$z\,\frac{f}{g} = \langle f\,z, g\,z\rangle$$

prop

$$\begin{tikzcd}
 & G \arrow[dl, "f"'] \arrow[d, "\frac{f}{g}"] \arrow[dr, "g"] & \\
E & E \oplus F \arrow[l, "fst"] \arrow[r, "snd"'] & F
\end{tikzcd}$$

example Let

$$\begin{tikzcd}
E \arrow[r, phantom, "\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}"] & G \arrow[l, "f"'] \arrow[r, "g"] \arrow[r, phantom, shift right=3, "\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}"] & F \\
2 & 3 & 4
\end{tikzcd}$$

Then

$$\begin{tikzcd}
G \arrow[d, "\frac{f}{g}"'] \arrow[r, phantom, shift left=3, "\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}"] \\
E \oplus F
\end{tikzcd}$$

Sum

define Let $E \xrightarrow{f} G \xleftarrow{g} F$.

We define

$$E \oplus F \xrightarrow{f|g} G$$
$$\langle x, y\rangle \rightsquigarrow f\,x + g\,y$$

We'll often write

$$\langle x, y\rangle\, f|g := (f|g)\,\langle x, y\rangle$$

So

$$\langle x, y\rangle\, f|g = f\,x + g\,y$$

prop

$$\begin{tikzcd}
E \arrow[r, "injl"] \arrow[dr, "f"'] & E \oplus F \arrow[d, "f|g"] & F \arrow[l, "injr"'] \arrow[dl, "g"] \\
 & G &
\end{tikzcd}$$

example Let

$$\begin{tikzcd}
E \arrow[r, phantom, shift left=3, "\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}"] & G \arrow[l, "f"'] \arrow[r, phantom, shift left=3, "\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}"] & F \arrow[l, "g"'] \\
2 & 3 & 4
\end{tikzcd}$$

Then

$$\begin{tikzcd}
E \oplus F \arrow[d, "f|g"'] \arrow[r, phantom, shift left=3, "\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}"] \\
G
\end{tikzcd}$$

define Let $\left[E \oplus F\right] \oplus G \xrightarrow{assoc} E \oplus \left[F \oplus G\right]$

$$\langle\langle x, y\rangle, z\rangle \rightsquigarrow \langle x, \langle y, z\rangle\rangle$$

prop Let

$$\begin{tikzcd}
E_1 \arrow[dr, "f"] & \\
E_2 \arrow[r, "g"] & F \\
E_3 \arrow[ur, "h"'] &
\end{tikzcd}$$

Then

$$\begin{tikzcd}
E_1 \arrow[r, "f"] \arrow[d, phantom, "\oplus"] & \\
E_2 \arrow[r, "g"] \arrow[d, phantom, "\oplus"] & F \\
E_3 \arrow[r, "h"] &
\end{tikzcd}
\;=\;
\begin{tikzcd}
E_1 \arrow[d, phantom, "\oplus"] \\
E_2 \arrow[d, phantom, "\oplus"] \\
E_3
\end{tikzcd}
\;\overset{assoc}{\cong}\;
\begin{tikzcd}
E_1 \arrow[r, "f"] \arrow[d, phantom, "\oplus"] & \\
E_2 \arrow[r, "g"] \arrow[d, phantom, "\oplus"] & F \\
E_3 \arrow[r, "h"] &
\end{tikzcd}$$

i.e. $(f \mid g) \mid h = f \mid (g \mid h)$, assoc

prop Let

$$\begin{tikzcd}
 & F_1 \\
E \arrow[r, "g"] \arrow[ur, "f"] \arrow[dr, "h"'] & F_2 \\
 & F_3
\end{tikzcd}$$

Then

$$\begin{tikzcd}
 & F_1 \arrow[d, phantom, "\oplus"] \\
 & F_2 \arrow[d, phantom, "\oplus"] \\
E \arrow[r, "g"] \arrow[ur, "f"] \arrow[dr, "h"'] & F_3 \\
 & F_3
\end{tikzcd}
\;\overset{assoc}{\cong}\;
\begin{tikzcd}
 & F_1 \arrow[d, phantom, "\oplus"] \\
 & F_2 \arrow[d, phantom, "\oplus"] \\
E \arrow[r, "g"] \arrow[ur, "f"] \arrow[dr, "h"'] & F_3 \\
 & F_3
\end{tikzcd}
\;=\;
\begin{tikzcd}
 & F_1 \arrow[d, phantom, "\oplus"] \\
 & F_2 \arrow[d, phantom, "\oplus"] \\
E \arrow[r, "g"] \arrow[ur, "f"] \arrow[dr, "h"'] & F_3 \\
 & F_3
\end{tikzcd}$$

i.e.

$$\text{assoc.}\ \begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

prop

$$\begin{tikzcd}
 & E_1 \arrow[dr, "f"] & \\
E \arrow[ur] \arrow[dr] & & F \\
 & E_2 \arrow[ur, "g"'] &
\end{tikzcd}
\;\rightsquigarrow\;
\begin{tikzcd}
E_1 \arrow[r, "f"] \arrow[d, phantom, "\oplus"] & F \\
E_2 \arrow[r, "g"] &
\end{tikzcd}$$

$$\begin{tikzcd}
E_1 \arrow[d, "f"] \\
E_2 \arrow[d, "g"] \\
E_3
\end{tikzcd}
\;\rightsquigarrow\;
\begin{tikzcd}
E_1 \arrow[r, "f"] \arrow[d, phantom, "\oplus"] & \\
E_2 \arrow[r, "g"] \arrow[d, phantom, "\oplus"] & \\
E_3 &
\end{tikzcd}$$

$$\begin{tikzcd}
E_1 \arrow[r, "f"] \arrow[d, phantom, "\oplus"] & \\
E_2 \arrow[r, "g"] \arrow[d, phantom, "\oplus"] & \\
E_3 &
\end{tikzcd}
\;\overset{assoc}{\cong}\;
\begin{tikzcd}
E_1 \arrow[r, "f"] \arrow[d, phantom, "\oplus"] & \\
E_2 \arrow[r, "g"] \arrow[d, phantom, "\oplus"] & \\
E_3 &
\end{tikzcd}
\;\rightsquigarrow\;
\begin{tikzcd}
E_1 \arrow[r, "f"] \arrow[d, phantom, "\oplus"] & \\
E_2 \arrow[r, "g"] \arrow[d, phantom, "\oplus"] & \\
E_3 &
\end{tikzcd}$$

prop

$$\begin{tikzcd}
 & F_1 \arrow[dr, "f"] & \\
E_1 \arrow[ur, "f"] \arrow[dr, "g"'] & & E_2 \\
 & F_2 \arrow[ur, "g"'] &
\end{tikzcd}
\;\rightsquigarrow\;
\begin{tikzcd}
E_1 \arrow[r, "f"] & F_1 \\
E_2 \arrow[r, "g"] & F_2
\end{tikzcd}
\;=\;
\begin{tikzcd}
E_1 \arrow[r, "f"] & F_1 \\
E_2 \arrow[r, "g"] & F_2
\end{tikzcd}$$

$$\begin{tikzcd}
E_1 \arrow[d, "f"] \\
E_2 \arrow[d, "g"] \\
E_3
\end{tikzcd}
\;\rightsquigarrow\;
\begin{tikzcd}
E_1 \arrow[r, "f"] & F_1 \\
E_2 \arrow[r, "g"] & F_2 \\
E_3 &
\end{tikzcd}
\;\overset{assoc}{\cong}\;
\begin{tikzcd}
E_1 \arrow[r, "f"] & F_1 \\
E_2 \arrow[r, "g"] & F_2 \\
E_3 &
\end{tikzcd}
\;\rightsquigarrow\;
\begin{tikzcd}
E_1 \arrow[r, "f"] & F_1 \\
E_2 \arrow[r, "g"] & F_2 \\
E_3 &
\end{tikzcd}$$

prop

$$\begin{tikzcd}
 & F_1 \arrow[dr, "f"] & \\
E \arrow[ur, "f"] \arrow[dr, "g"'] & & E \\
 & F_2 \arrow[ur, "g"'] &
\end{tikzcd}
\;\rightsquigarrow\;
\begin{tikzcd}
E \arrow[r, "f"] & F_1 \\
E \arrow[r, "g"] & F_2
\end{tikzcd}
\;=\;
\begin{tikzcd}
E \arrow[r, "f"] & F_1 \\
E \arrow[r, "g"] & F_2
\end{tikzcd}$$

$$\begin{tikzcd}
E \arrow[d, "f"] \\
E \arrow[d, "g"] \\
E
\end{tikzcd}
\;\rightsquigarrow\;
\begin{tikzcd}
E \arrow[r, "f"] & F_1 \\
E \arrow[r, "g"] & F_2 \\
E &
\end{tikzcd}
\;\overset{assoc}{\cong}\;
\begin{tikzcd}
E \arrow[r, "f"] & F_1 \\
E \arrow[r, "g"] & F_2 \\
E &
\end{tikzcd}
\;\rightsquigarrow\;
\begin{tikzcd}
E \arrow[r, "f"] & F_1 \\
E \arrow[r, "g"] & F_2 \\
E