Lattice of Subspaces of Enclidean Space

Let H be a finite-dimensional $\mathbb{R}$-vector space [visualize $\mathbb{R}^3$]
with an inner-product
$$\langle\cdot|\cdot\rangle : H\times H \to \mathbb{R}$$
Recall the basic definition
define Given $v, w : H$, say
v is orthogonal to w iff $\langle v|w\rangle = 0$,
Write $v \perp w$ ■

We'll consider the lattice of subspaces
$$\mathrm{Sub}[H] := \{a \subseteq H \mid a \text{ is a linear subspace}\} \text{ ordered by } \subseteq$$
and abstract out a few of its essential features.

First, recall that
$\forall a, b : \mathrm{Sub}[H]$,
meet $a \wedge b = a \cap b$
& join $a \vee b$ = the subspace generated by a & b.

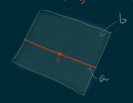
We'll simply write $a \leq b$ for $a \subseteq b$.
Lattice $\mathrm{Sub}[H]$ has bottom $0 := \{0 : H\}$
& top $1 := H$ elements.

Since we've equipped H with an inner-product,
we can define the orthogonality relation on $\mathrm{Sub}[H]$.
define $\forall a, b : \mathrm{Sub}[H]$,
$a \perp b$ iff $\forall v : a,\ w : b.\ v \perp w$ ■,
and we can define the orthocomplement of a subspace
define $\forall a : \mathrm{Sub}[H]$,
$a^\perp := \{w : H \mid \forall v : a.\ v \perp w\}$ ■

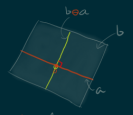
The operation $(\cdot)^\perp : \mathrm{Sub}[H] \longrightarrow \mathrm{Sub}[H] : \mathrm{Set}$
$a \rightsquigarrow a^\perp$
is an involution

define Let P be a poset. Then a map
$(\ )^* : P \longrightarrow P : \mathrm{Set}$
is an involution
iff
(comat) $\forall x, y : P$
if $x \leq y$ then $x^* \geq y^*$
(idempot) $\forall x : P$
$x^{**} = x$. [i.e. it is its own inverse $(\cdot)^* \circ (\cdot)^* = id_P$]
We'll write $(\ )^* : P^{op} \longrightarrow P : \mathrm{Poset}$ ■
Involutions have an important property which is easy to prove
prop Let $P : \mathrm{Poset}$ with finite joins $\vee$.
Let $(\ )^* : P^{op} \longrightarrow P$ be an involution
Then P has finite meets $\wedge$ &
$a \wedge b = (a^* \vee b^*)^*$,
$1 = 0^*$
Also DeMorgan Laws hold
$(a \vee b)^* = a^* \wedge b^*$,
$(a \wedge b)^* = a^* \vee b^*$. ■
The statement holds [for the same reasons]
for arbitrary [not necessarily finite] joins.

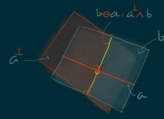
The orthocomplement $(\ )^\perp : \mathrm{Sub}[H]^{op} \longrightarrow \mathrm{Sub}[H]$
plays an important role in $\mathrm{Sub}[H]$.
Let $a \leq b : \mathrm{Sub}[H]$, visualizing in $\mathbb{R}^3$:



Then it makes sense to subtract a from b denoted $b \ominus a$.



By subtraction I mean
$a \vee (b \ominus a) = b$.
This subtraction can always be computed
with the help of the orthocomplement as follows:



$$b \ominus a = a^\perp \wedge b.$$

Note $a \perp b \ominus a$
& $a \vee (b \ominus a) = b$,
i.e. b is the direct sum of a & $a^\perp \wedge b$.

Abstracting this situation out we get to the notion of

Orthomodular Lattice

define An orthomodular lattice consists of
$A, \vee, \wedge, 0, 1$ a bounded lattice
with $(\ )^\perp : A \longrightarrow A$ an involution
where (orthomodularity)
$\forall a \leq b : A$,
$a \vee (a^\perp \wedge b) = b$ ■

Now we'll abstract further.
Recall that $\mathrm{Sub}[H]$ had the orthogonality relation $\perp$.
In $\mathrm{Sub}[H]$ it can be characterized purely
in terms of orthomodular lattice structure:
given $a, b : \mathrm{Sub}[H]$, then
$a \perp b$ iff $a \leq b^\perp$.
This leads us to
define Let A be orthomodular lattice, $a, b : A$.
Say
a is orthogonal to b iff $a \leq b^\perp$.
Write $a \perp b$ ■
example (1) Let $a : A$. Then
$a \perp a^\perp$
since $a \leq a^{\perp\perp} = a$.
Also note $a \vee a^\perp = 1$.
In this case we could suggestively write
$a \oplus a^\perp = 1$.
(2) Let $a \leq b : A$. Then
$a \perp a^\perp \wedge b$
since $a \leq (a^\perp \wedge b)^\perp = a \wedge b^\perp$.
Also note $a \vee (a^\perp \wedge b) = b$,
so we could write
$a \oplus (a^\perp \wedge b) = b$. ■

define $A \oplus_\perp A := \{\langle a, b\rangle : A \times A \mid a \perp b\}$ ■

Then we can think of direct sum $\oplus$
as a partial binary operation on A
whose domain of definition is
$A \oplus_\perp A$,
i.e. $\oplus : A \oplus_\perp A \longrightarrow A$.

We'll abstract the relation $\perp$ & partial operation $\oplus$.
But what properties of $\perp$ should we abstract?
symmetry $a \perp b \Rightarrow b \perp a$
& orthogonality to 0 $a \perp 0$, $0 \perp a$ seem obvious.
Orthogonality isn't reflexive nor transitive...
Amazingly the hint is given to us by the direct sum.
Note that $\oplus$ is a restriction of $\vee : A \times A \longrightarrow A$
which is associative, commutative, & 0 is its neutral element.
To abstract the properties of $\oplus$ it would be nice to
say that it is somehow
associative, commutative, & 0 is its neutral element.
But to even say that
$a \oplus (b \oplus c) = (a \oplus b) \oplus c$
we require for $a \oplus (b \oplus c)$ to be defined, i.e.
we require $b \perp c$, $a \perp b \oplus c$;
then we could state that $a \perp b$, $a \oplus b \perp c$.
Which leads to

Partial Commutative Monoid

define Let $M : \mathrm{Set}$ with a binary relation $\perp$ on M.
Let $M \oplus_\perp M := \{\langle a, b\rangle : M \times M \mid a \perp b\}$.
Then a PCM [Partial Commutative Monoid] on $M, \perp$ consists of
$\oplus : M \oplus_\perp M \longrightarrow M$
$0 : M$
where
(assoc) $\forall a, b, c : M$,
if $b \perp c$, $a \perp b \oplus c$
then $a \perp b$, $a \oplus b \perp c$
$a \oplus (b \oplus c) = (a \oplus b) \oplus c$
(comm) $\forall a, b : M$,
if $a \perp b$
then $b \perp a$,
$a \oplus b = b \oplus a$
(neutr) $\forall a : M$,
$a \perp 0$, $0 \perp a$,
$a \oplus 0 = a = 0 \oplus a$ ■

Note that any binary associative operation can be lifted
to operate on any finite number of inputs, i.e.
given $\times : M \times M \longrightarrow M$ be assoc with neutral 1
Then we have $\mathrm{List}[M] \longrightarrow M$
$() \rightsquigarrow 1$
$a \rightsquigarrow a$
$(a, b) \rightsquigarrow a \times b$
$(a, b, c) \rightsquigarrow a \times b \times c$
$\vdots$
We can do something similar for the partial $\oplus$.
define $M \oplus_\perp M \oplus_\perp M := \{\langle a, b, c\rangle : M^{\times 3} \mid a \perp b,\ a \perp c,\ b \perp c,\ a \perp b \oplus c,\ b \perp a \oplus c,\ c \perp a \oplus b\}$ ■
Symmetry of $\perp$ gives us $\mathbb{S}^3$ group action on $M \oplus_\perp M \oplus_\perp M$
We immediately get
prop $\langle a, b, c\rangle : M \oplus_\perp M \oplus_\perp M$ iff $b \perp c$, $a \perp b \oplus c$
iff $a \perp c$, $b \perp a \oplus c$
iff $a \perp b$, $c \perp a \oplus b$ ■
So we can extend $\oplus$ to
$\oplus : M \oplus_\perp M \oplus_\perp M \longrightarrow M$
and so on for higher arities.

define Let M, N be PCMs.

Then a morphism from M to N of PCMs consists of

$$f : M \longrightarrow N : \mathrm{Set}$$

where (1) $f(0) = 0$

(2) $\forall a, b : M.$
$\quad$ if $a \perp b$
$\quad$ then $f(a) \perp f(b)$
$\qquad f(a \oplus b) = f(a) \oplus f(b)$ ■

This gives us a category PCM.

Now we'll see what we can express in the language of PCMs

define Let $M : \mathrm{PCM}$ with $\perp, \oplus$.

Define a relation $\leq$ on M

$\forall a, b : M.$
$\quad a \leq b$
$\quad$:iff $\exists x : M.$
$\qquad a \perp x,$
$\qquad x \oplus a = b$ ■

prop $\leq$ is a preorder.

i.e. $\leq$ is reflexive & transitive

proof (reflexivity) Note $a \perp 0$, $a \oplus 0 = a$.

(transitivity) Let $a \leq b$, $b \leq c$.

Then we have $x, y : A$ s.t.

$$a \perp x, \quad x \oplus a = b$$
$$b \perp y, \quad y \oplus b = c.$$

So $c = y \oplus b$
$\quad = y \oplus (x \oplus a), \quad y \perp x$
$\quad = (y \oplus x) \oplus a.$

Hence $a \leq c$. ■

q Is M a poset? No... we are unable to prove antisymmetry for general PCMs. We'll return to this later.

We have actually shown something more interesting:

prop M is a category

(obj) whose objects are elements of M

(morphism) & given $a, b : M$, a morphism

$$a \xrightarrow{x} b : M$$

is an element $x : M$ s.t.

$$x \perp a,$$
$$a \oplus x = b.$$

(composition) Given morphisms $a \xrightarrow{x} b \xrightarrow{y} c : M$, their composition is

$$a \xrightarrow{x \oplus y} c : M$$

(identity) identity morphism is

$$a \xrightarrow{0} a : M.$$ ■

We will have no need to study this category explicitly, because for the PCMs we will consider there will be almost one morphism between any two objects, i.e. it will be a preorder.

But we will borrow the morphism notation, which will simplify our proofs.

notation Write $a \xrightarrow{x} b$ or $a \overset{x}{\leq} b$

for $\quad x \perp a$
& $\quad a \oplus x = b$. ■

We have the following elementary properties

prop (refl) $a \overset{0}{\leq} a$

(trans) $a \overset{x}{\leq} b \overset{y}{\leq} c \Rightarrow x \perp y,\ a \overset{x \oplus y}{\leq} c.$

(pos) $\forall a \perp b : M.$

$$a \overset{b}{\leq} a \oplus b$$
$$b \overset{a}{\leq} a \oplus b$$

(sym) $a_1 \overset{a_2}{\leq} b \Rightarrow a_2 \overset{a_1}{\leq} b$

(bottom) $0 \overset{a}{\leq} a$ ■

prop Let $M \xrightarrow{f} N : \mathrm{PCM}$, then

$\forall a, b : M.$
$\quad$ if $a \overset{k}{\leq} b$
$\quad$ then $f(a) \overset{f(k)}{\leq} f(b)$ (monotonicity)

proof Let $a \overset{k}{\leq} b$. Then

$$a \perp k, \quad a \oplus k = b,$$

so $f(a) \perp f(k)$, $\quad f(a) \oplus f(k) = f(b)$,

i.e. $f(a) \overset{f(k)}{\leq} f(b)$ ■

example Let A be an orthomodular lattice with $\vee, \wedge, 0, 1, (\cdot)^{\perp}$.

Define orthogonality

$$a \perp b \ :\text{iff}\ a \leq b^{\perp},$$
$$a \oplus b := a \vee b$$

prop A is a PCM with $\perp, \oplus, 0$

proof It's enough to show the properties of $\perp$ related to (associativity).

Let $a, b, c : A$ s.t. $b \perp c$, $a \perp b \oplus c$.

goal $a \perp b$.

Note $a \leq (b \vee c)^{\perp} = b^{\perp} \wedge c^{\perp}$.

hence $a \leq b^{\perp}$, i.e. $a \perp b$.

goal $a \oplus b \perp c$, i.e. $a \vee b \leq c^{\perp}$.

So it's enough to show $a \leq c^{\perp}$, $b \leq c^{\perp}$.

Note $a \leq (b \vee c)^{\perp} = b^{\perp} \wedge c^{\perp}$,

so $a \leq c^{\perp}$.

And we assumed $b \leq c^{\perp}$. ■

Such PCMs have very special properties like

prop $\forall a : A.$
$\quad$ if $a \perp a$ then $a = 0$

proof Let $a \perp a$, then $a \leq a^{\perp}$.

Note $a \oplus a^{\perp} = a \vee a^{\perp} = 1$,
$\qquad\qquad\qquad\quad \overset{\shortparallel}{a^{\perp}}$.

So $a = 0$. ■

prop Let $a_1, a_2 \leq b : A$, $a_1 \perp a_2$.

Then $a_1 \oplus a_2 \leq b$. ■

We have another important & very different example.

example The interval $[0;1] \subseteq \mathbb{R}$ is a PCM, where

(1) $\forall a, b : [0;1].$
$\quad a \perp b$:iff $a + b : [0;1]$

($\oplus$) $\forall a \perp b : [0;1].$
$\quad a \oplus b := a + b.$

Consider arbitrary $p : M \longrightarrow [0;1] : \mathrm{PCM}$.

If we see elements of M as events in probability space, then p is like a probability function, i.e.

$$p(0) = 0 : [0;1]$$
$$\forall e, e' : M.$$
$$\text{if } \underset{\text{disjoint}}{e \perp e'} \text{ then } p(e \oplus e') = p(e) + p(e') : [0;1]$$

example Every commutative monoid is a partial commutative monoid where any two elements are orthogonal.

So especially every commutative group $A, +, -, 0$ is a PCM.

q What is the preorder?

Let $x, y : A$. Then $x + (y - x) = y$,

hence $x \leq y$.

So the preorder is trivial. ■

example Consider the interval $[0, \infty) \subseteq \mathbb{R}$.

Then $[0, \infty)$ is a commutative monoid with $+, 0$.

If we consider it as a PCM with trivial orthogonality, then the PCM preorder coincides with the standard ordering

i.e. $a \leq b$ iff $\exists k : [0, \infty).$
$\qquad a + k = b$. ■

Now we define for our purposes the most important subsets of PCMs

define $\forall a : M.$

$$\perp(a) := \{ x : M \mid x \perp a \}$$
$$\mathrm{down}(a) := \{ x : M \mid x \leq a \}$$
$$\mathrm{up}(a) := \{ x : M \mid x \geq a \}$$ ■

Obviously for $f : M \longrightarrow N : \mathrm{PCM}$ we have

$$\perp(a) \xrightarrow{f} \perp(f(a))$$
$$\mathrm{down}(a) \xrightarrow{f} \mathrm{down}(f(a))$$
$$\mathrm{up}(a) \xrightarrow{f} \mathrm{up}(f(a))$$

prop Let $a' \leq a$, $a \perp b$

Then $a' \perp b$

proof Let $a' \overset{x}{\leq} a$, then

$$a \oplus b = (a' \oplus x) \oplus b,$$

& $a' \perp x$, $a' \oplus x \perp b$.

So $a' \perp x$. ■

prop (1) $\perp \subseteq M^{op} \times M$

i.e. $\forall x' \leq x \perp y \geq y' : M.$
$\quad x' \perp y'$

(2) $\perp(a)$, $\mathrm{down}(a)$ are downward closed,

$\mathrm{up}(a)$ is upwards closed.

q. $\perp(a)$ feels like a downward closure of something.

Is there an element b representing $\perp(a)$?

i.e. $\perp(a) = \mathrm{down}(b)$?

We'll get back to this question later.

Already we can show the following.

prop Let $a : M$. Then we have a surjective monotone map

$$a \oplus \cdot : \perp(a) \longrightarrow \mathrm{up}(a)$$
$$b \rightsquigarrow a \oplus b,$$

& $\forall b, c : \perp(a).$

$$\text{if} \quad b \overset{x}{\leq} c$$
$$\text{then } a \oplus b \overset{x}{\leq} a \oplus c.$$

proof The type & surjectivity of $a \oplus \cdot$ is obvious.

Let $b \overset{x}{\leq} c$ (monotonicity). Then

$$a \oplus c = a \oplus (b \oplus x),\ x \perp a \oplus b$$
$$= (a \oplus b) \oplus x.$$ ■

Also note

$$a \oplus \cdot : \perp(a) \longrightarrow \mathrm{up}(a)$$
$$0 \rightsquigarrow a$$

As we will see, properties of maps such as $a \oplus \cdot$, control the structure of M.

Subtraction

Let $a \overset{x}{\leq} b$. Then $a \oplus x = b$. So we can think of x as $b - a$. But there may be many such x's. So we may not speak of some global canonical subtraction.

Note that requiring uniqueness of subtraction for all pairs $a \leq b$, is the same as requiring

$$\forall a : M.$$
$$a \oplus \cdot : \bot(a) \longrightarrow up(a)$$

i.e. to be injective

which is the same as

define $M : PCM$ satisfies cancelativity

:iff $\forall a, x, y : E.$

if $a \perp x, y,$

$a \oplus x = a \oplus y$

then $x = y$. ■

Note that cancelativity can be restated:

$\forall a \leq b : E.$

if $a \overset{x}{\leq} b,\ a \overset{y}{\leq} b$

then $x = y$

This is precisely that M must be a preorder category.

Let M satisfy cancelativity.

Then given $a \leq b$, we have unique element $b \ominus a : M$

s.t. $a \perp b \ominus a, \quad a \oplus (b \ominus a) = b$

i.e. $a \overset{b \ominus a}{\leq} b$.

So we get a partial binary operation on M

$$\ominus : \{\langle b, a \rangle : M \times M \mid a \leq b\} \longrightarrow M.$$

prop Let $f : M \to N : PCM$, M, N satisfy cancelativity.

Then $\forall a \leq b : M.$

$$f(b \ominus a) = f(b) \ominus f(a)$$

proof Immediate from monotonicity

$$a \overset{\ell}{\leq} b \Rightarrow f(a) \overset{f(\ell)}{\leq} f(b). \ ■$$

Let M satisfy cancelativity, $a : M$; then

$$a \oplus \cdot : \bot(a) \longrightarrow up(a)$$
$$x \rightsquigarrow a \oplus x$$

is a bijection [it is injective & surjective]

& its inverse is

$$\cdot \ominus a : up(a) \longrightarrow \bot(a).$$
$$x \rightsquigarrow x \ominus a$$

This means

prop (1) $\forall x \perp a : M.$

$$(a \oplus x) \ominus a = x$$

(2) $\forall x \geq a : M.$

$$a \oplus (x \ominus a) = x \ ■$$

prop $\cdot \ominus a : up(a) \longrightarrow \bot(a)$

is monotone.

more precisely,

$\forall x, y : up(a).$

$$x \overset{\ell}{\leq} y \Rightarrow x \ominus a \overset{\ell}{\leq} y \ominus a$$

proof Let $a \overset{\alpha}{\leq} x,\ a \overset{\beta}{\leq} y,\quad x \overset{\ell}{\leq} y.$

Then $a \overset{\alpha}{\leq} x,\ x \overset{\ell}{\leq} y.$

$a \overset{\alpha \oplus \ell}{\leq} y$, but we also have $a \overset{\beta}{\leq} y$,

so by cancelativity

$$\alpha \oplus \ell = \beta,$$

i.e.

$$\underset{x \ominus a}{\alpha} \overset{\ell}{\leq} \underset{y \ominus a}{\beta}. \ ■$$

Now we'll look at

$$a \ominus \cdot : down(a) \longrightarrow down(a)$$
$$x \rightsquigarrow a \ominus x.$$

prop $a \ominus \cdot : down(a) \longrightarrow down(a)$

(1) is its own inverse

(2) is comonotone & more precisely

$$\forall x, y : down(a).$$
$$x \overset{\ell}{\leq} y \Rightarrow a \ominus y \overset{\ell}{\leq} a \ominus x$$

proof (1) Let $x \overset{k}{\leq} a$. Then $k \overset{x}{\leq} a$,

but $k = a \ominus x$ so

$$x = a \ominus k = a \ominus (a \ominus x)$$

(2) Let $x \overset{\alpha}{\leq} a,\ y \overset{\beta}{\leq} a,\ x \overset{\ell}{\leq} y.$

Then $x \overset{\ell}{\leq} y,\ y \overset{\beta}{\leq} a,$

$x \overset{\ell \oplus \beta}{\leq} a$, & $x \overset{\alpha}{\leq} a$

so by cancelativity

$$\ell \oplus \beta = \alpha,$$

i.e.

$$\underset{a \ominus y}{\beta} \overset{\ell}{\leq} \underset{a \ominus x}{\alpha} \ ■$$

We can think of $a \ominus \cdot$ as relative orthocomplement in $down(a)$.

[beware, $down(a)$ need not be closed under $\oplus$]

Because of that we introduce the notation

Given $u : M$,

$$\forall a : down(u).$$
$$a^{\perp_u} := u \ominus a.$$

In this notation

prop (1) $\forall a : down(u).$

$a^{\perp_u} : down(u)$ is unique with the property

$$a \perp a^{\perp_u},$$
$$a \oplus a^{\perp_u} = u$$

(2) $\forall a : down(u).\ a^{\perp_u \perp_u} = a,$

(3) $\forall a, b : down(u).$

$$a \overset{\ell}{\leq} b \Rightarrow b^{\perp_u} \overset{\ell}{\leq} a^{\perp_u}$$

To summarize, for $M : PCM$ having cancelativity is the same as having the following pairs of mutual monotone isomorphisms

$\forall a : M.$

$$\bot(a) \underset{\cdot \ominus a}{\overset{a \oplus \cdot}{\rightleftarrows}} up(a)$$

$$down(a) \underset{a \ominus \cdot}{\overset{a \ominus \cdot}{\underset{op}{\rightleftarrows}}} down(a)$$

involution

Now let's see what happens when we assume antisymmetry on $M : PCM$

Let $a \overset{b}{\leq} 0 : M$, i.e. $a \perp b$, $a \oplus b = 0$.

We obviously have $0 \overset{a}{\leq} a$.

Hence if we had antisymmetry, we'd get

$$a = 0, \quad \text{so} \quad \underset{b}{0 \oplus b} = 0.$$

define M satisfies positivity

:iff $\forall a, b : M.$

if $a \perp b,$

$a \oplus b = 0$

then $a = 0 = b$. ■

If we require both positivity & cancelativity, we recover antisymmetry:

let $a, b : M,\quad a \overset{x}{\leq} b,\quad b \overset{y}{\leq} a,$

then $a \overset{x \oplus y}{\leq} a$, but $a \overset{0}{\leq} a$,

so by cancelativity

$$x \oplus y = 0,$$

& by positivity

$$x = 0 = y,$$

so $\quad a = b$, since $a \oplus x = b$.

Augmenting cancelativity with positivity won't change the theory all that much — but it will serve as a filter to some perhaps undesirable examples such as $\mathbb{R}$ with trivial orthogonality & the usual addition.

This motivates the following

define A generalized effect algebra consists of

$M : PCM$ satisfying cancelativity & positivity.

We denote by $GEff$ the full-subcategory of PCM ■

define An effect algebra consists of

$E : GEff$ with $1 \in E$ top element.

We denote by Eff the full-subcategory of PCM ■

These are very miserable naming conventions — a better choice would be

generalized effect algebra $\rightsquigarrow$ effect algebra

effect algebra $\rightsquigarrow$ bounded effect algebra

or even

generalized effect algebra $\rightsquigarrow$ ring of effects

effect algebra $\rightsquigarrow$ field of effects

by analogy from measure theory

but it seems to be too late.

Note that a generalized effect algebra is the same thing as a partial commutative monoid whose underlying category is a poset; This seems to make the notion of a generalized effect algebra very natural.

Orthocomplement & Effect Algebras

Recall that we arrived at PCMs by abstracting out the direct sum aspects of Orthomodular Lattices; but in the process we lost the orthocomplement.

This orthocomplement aspect is recovered in Effect Algebra.

Let $a : E : \mathbf{Eff}$, since E has top element 1, we always have

$$a \overset{1 \ominus a}{\leq} 1,$$

i.e. $a \perp 1 \ominus a$, $\quad a \oplus (1 \ominus a) = 1$.

define Let $E : \mathbf{Eff}$; def.

$$(\ )^{\perp} : E \longrightarrow E$$

$$a^{\perp} := 1 \ominus a \ \blacksquare$$

Since $(\ )^{\perp} = 1 \ominus \cdot : \underset{E}{\mathrm{dom}(1)} \longrightarrow \underset{E}{\mathrm{dom}(1)}$, we immediately get

prop (1) $\forall a : E$.

$a^{\perp} : E$ is unique with the property

$$a \perp a^{\perp},$$

$$a \oplus a^{\perp} = 1$$

(2) $\forall a : E$. $\quad a^{\perp\perp} = a$,

(3) $\forall a, b : E$. $\quad a \overset{\mathbf{E}}{\leq} b \Rightarrow b^{\perp} \overset{\mathbf{E}}{\leq} a^{\perp}$. $\blacksquare$

prop $\forall a : E$.

if $a \perp 1$ then $a = 0$

i.e.

$\perp(1) = \{0\}$

proof Let $a \perp 1$. Then

$$(a \oplus 1) \oplus (a \oplus 1)^{\perp} = 1$$

$$\overset{\shortparallel}{1 \oplus (a \oplus (a \oplus 1)^{\perp})} = 1,$$

so $a \oplus (a \oplus 1)^{\perp} = 0$, so $a = 0$. $\blacksquare$

Finally

prop $\perp(a) = \mathrm{down}(a^{\perp})$

proof $\subseteq$: Let $x \perp a$. Then

$$(x \oplus a) \oplus (x \oplus a)^{\perp} = 1,$$

$$a \oplus (x \oplus (x \oplus a)^{\perp}),$$

so $x \oplus (x \oplus a)^{\perp} = a^{\perp}$,

i.e. $x \overset{\mathbf{E}}{\leq} a^{\perp}$.

$\supseteq$: Let $x \overset{\mathbf{E}}{\leq} a^{\perp}$. Then

$$x \oplus b = a^{\perp}.$$

Note $\quad a \oplus a^{\perp} = 1$

$$a \oplus (x \oplus b),$$

so $a \perp x$. $\blacksquare$

Lattice Properties

Now we'll see what happens when we add lattice structure.

Recall a very easy theorem

prop Let $P, Q : \mathbf{Pos}$, & $P \xrightarrow{f} Q$ monotonic isomorphism (i.e. inverse is also monotonic).

Let $x_{\bullet} : I \longrightarrow P$ be a family of elements of P.

Then if $\bigvee_{i : I} x_i$ exists in P

then $\bigvee_{i : I} f(x_i)$ exists in Q,

& $f\left(\bigvee_{i : I} x_i\right) = \bigvee_{i : I} f(x_i) : Q$.

Similarly for meets $\bigwedge$. $\blacksquare$

Let $M : \mathbf{GEff}$, $a : M$. Then, since

$$\begin{tikzcd} \perp(a) \arrow[r, shift left, "a \oplus \cdot"] & \mathrm{up}(a) \arrow[l, shift left, "\cdot \ominus a"] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{down}(a) \arrow[r, shift left, "a \ominus \cdot"] & \mathrm{down}(a) \arrow[l, shift left, "a \ominus \cdot"] \end{tikzcd}^{op}$$

are monotonic maps,

they preserve all meets & joins in $\perp(a)$, $\mathrm{down}(a)$, $\mathrm{up}(a)$.

Q What are meets & joins in $\perp(a)$, $\mathrm{down}(a)$, $\mathrm{up}(a)$?

Recall that $\bigvee \emptyset$ is the bottom element,

& $\bigwedge \emptyset$ is the top element

$\perp(a)$ & $\mathrm{down}(a)$ are closed under all joins $\bigvee$ of M,

since they are downward closed;

It is easy to see that they are closed under

all non-empty meets $\bigwedge$ of M.

Similarly $\mathrm{up}(a)$ is closed under all meets $\bigwedge$

& all non-empty joins of M.

Immediately we get the following

prop Let $M : \mathbf{GEff}$ be a complete lattice with respect to $\leq$,

let $a : M$. Then

(1) $\forall I : \mathbf{Set}$, $b_{\bullet} : I \longrightarrow \perp(a)$.

$$a \oplus \bigvee_i b_i = \bigvee_i (a \oplus b_i)$$

(2) $\forall I : \mathbf{Set}$ non-empty, $b_{\bullet} : I \longrightarrow \perp(a)$.

$$a \oplus \bigwedge_i b_i = \bigwedge_i (a \oplus b_i)$$

(3) $\forall I : \mathbf{Set}$ non-empty, $b_{\bullet} : I \longrightarrow \mathrm{up}(a)$.

$$\left[\bigvee_i b_i\right] \ominus a = \bigvee_i [b_i \ominus a]$$

(4) $\forall I : \mathbf{Set}$, $b_{\bullet} : I \longrightarrow \mathrm{up}(a)$.

$$\left[\bigwedge_i b_i\right] \ominus a = \bigwedge_i [b_i \ominus a]$$

(5) $\forall I : \mathbf{Set}$ non-empty, $b_{\bullet} : I \longrightarrow \perp(a)$.

$$a \ominus \bigvee_i b_i = \bigwedge_i (a \ominus b_i)$$

$$a \ominus \bigwedge_i b_i = \bigvee_i (a \ominus b_i)$$

prop Let $a_{\bullet}, b_{\bullet} : I \longrightarrow M$ s.t.

$\forall i : I$.

$$a_i \perp b_i$$

Then

$$\bigvee_{i : I} a_i \perp \bigwedge_{j : I} b_j$$

i.e. $\perp \leq M^{op} \times M$ respects lattice structure

proof Let $b_{\bullet} : I$. Then

$\forall i : I$

$$a_i \perp b_j \geq \bigwedge_{j : I} b_j,$$

so $a_i \perp \bigwedge_{j : I} b_j$,

i.e. $a_i : \perp\left(\bigwedge_{j : I} b_j\right)$.

But $\perp\left(\bigwedge_{j : I} b_j\right)$ is downward closed, so

$$\bigvee_{i : I} a_i : \perp\left(\bigwedge_{j : I} b_j\right). \ \blacksquare$$

Summary

$$a \overset{0}{\leq} a \quad \simeq \quad a \ominus a = 0$$

$$a \overset{x}{\leq} b \overset{y}{\leq} c \Rightarrow a \overset{x \oplus y}{\leq} c \quad \simeq \quad a \leq b \leq c \Rightarrow (c \ominus a) = (c \ominus b) \oplus (b \ominus a)$$

$$a \overset{x}{\leq} b \Rightarrow x \overset{a}{\leq} b \quad \simeq \quad b \ominus (b \ominus a) = a$$

$$a \overset{x,y}{\leq} b \Rightarrow x = y$$

$$\begin{tikzcd} \bot(a) \arrow[r, shift left, "a \oplus \cdot"] & up(a) \arrow[l, shift left, "\cdot \ominus a"] \end{tikzcd} \simeq$$

are mutual isomorphisms

$$\Downarrow$$

$$a \oplus (x \vee y) = (a \oplus x) \vee (a \oplus y)$$
$$a \oplus (x \wedge y) = (a \oplus x) \wedge (a \oplus y)$$

$$(x \vee y) \ominus a = (x \ominus a) \vee (y \ominus a)$$
$$(x \wedge y) \ominus a = (x \ominus a) \wedge (y \ominus a)$$

$$\forall b \overset{x}{\leq} c : \bot(a). \quad a \oplus b \overset{x}{\leq} a \oplus c. \quad \simeq \quad \forall b \leq c : \bot(a). \quad (a \oplus c) \ominus (a \oplus b) = c \ominus b$$

$$\forall b \overset{x}{\leq} c : up(a). \quad b \ominus a \overset{x}{\leq} c \ominus a \quad \simeq \quad \forall b \leq c : up(a). \quad (c \ominus a) \ominus (b \ominus a) = c \ominus b$$

$$(a \oplus x) \ominus a = x$$
$$a \oplus (x \ominus a) = x$$

$$\begin{tikzcd} down(a)^{op} \arrow[r, shift left, "a \ominus \cdot"] & down(a) \arrow[l, shift left, "a \ominus \cdot"] \end{tikzcd} \simeq$$

are mutual isomorphisms

$$\Downarrow$$

$$a \ominus (x \vee y) = (a \ominus x) \wedge (a \ominus y)$$
$$a \ominus (x \wedge y) = (a \ominus x) \vee (a \ominus y)$$

$$(x \vee y)^{\bot} = x^{\bot} \wedge y^{\bot}$$
$$(x \wedge y)^{\bot} = x^{\bot} \vee y^{\bot}$$

$$\forall b \overset{x}{\leq} c : down(a). \quad a \ominus c \overset{x}{\leq} a \ominus b \quad \simeq \quad (a \ominus b) \ominus (a \ominus c) = c \ominus b$$

$$a \ominus (a \ominus x) = x$$

$$\forall b \overset{x}{\leq} c. \quad c^{\bot} \overset{x}{\leq} b^{\bot} \quad \simeq \quad b^{\bot} \ominus c^{\bot} = c \ominus b$$

$$x^{\bot\bot} = x$$

Some trivial consequences

prop

(i) Let $a, b \overset{x,y}{\leq} s$. Then $a \vee b \overset{x \wedge y}{\leq} s$

$$\begin{tikzcd} a \arrow[dr, "a \vee b \ominus a"] \arrow[drr, bend left, "x"] & & \\ & a \vee b \arrow[r, "x \wedge y"] & s \\ b \arrow[ur, "a \vee b \ominus b"] \arrow[urr, bend right, "y"'] & & \end{tikzcd}$$

This follows from $s \ominus (a \vee b) = (s \ominus a) \wedge (s \ominus b)$

(i) Let $s \overset{x,y}{\leq} a, b$. Then $s \overset{x \vee y}{\leq} a \wedge b$

$$\begin{tikzcd} & & a \\ s \arrow[urr, bend left, "x"] \arrow[r, "x \wedge y"] \arrow[drr, bend right, "y"'] & a \wedge b \arrow[ur, "a \ominus (a \wedge b)"] \arrow[dr, "b \ominus (a \wedge b)"'] & \\ & & b \end{tikzcd}$$

This follows from $(a \wedge b) \ominus s = (a \ominus s) \wedge (b \ominus s)$

(ii) Let $a \vee b \overset{z}{\leq} s$. Then $a \overset{z \ominus (a \vee b \ominus a)}{\leq} s$, $b \overset{z \ominus (a \vee b \ominus b)}{\leq} s$

$$\begin{tikzcd} a \arrow[dr, "a \vee b \ominus a"] \arrow[drr, bend left, "(a \vee b) \ominus a \oplus z = s \ominus a"] & & \\ & a \vee b \arrow[r, "z"] & s \\ b \arrow[ur, "a \vee b \ominus b"] \arrow[urr, bend right, "(a \vee b) \ominus b \oplus z = s \ominus b"'] & & \end{tikzcd}$$

(ii) Let $s \overset{z}{\leq} a \wedge b$. Then $s \overset{z \oplus (b \ominus a \wedge b)}{\leq} a$, $s \overset{z \oplus (a \ominus a \wedge b)}{\leq} b$.

$$\begin{tikzcd} & & a \\ s \arrow[urr, bend left, "z \oplus (a \ominus a \wedge b) = a \ominus z"] \arrow[r, "z"] \arrow[drr, bend right, "z \oplus (b \ominus a \wedge b) = b \ominus s"'] & a \wedge b \arrow[ur, "a \ominus (a \wedge b)"] \arrow[dr, "b \ominus (a \wedge b)"'] & \\ & & b \end{tikzcd}$$

prop Let $a \perp b$, then

$$\begin{tikzcd} a \arrow[dr] \arrow[drr, bend left, "b"] & & \\ & a \vee b \arrow[r, dashed, "a \wedge b"] & a \oplus b \\ b \arrow[ur] \arrow[urr, bend right, "a"'] & & \end{tikzcd}$$

i.e. $a \oplus b = (a \vee b) \oplus (a \wedge b)$

$(a \oplus b) \ominus (a \vee b) = a \wedge b$ ∎

prop Let $a \leq b$, then

$$\begin{tikzcd} a \arrow[dr] \arrow[drr, bend left, "b \ominus a"] & & \\ & a \vee (b \ominus a) \arrow[r, "(b \ominus a) \wedge a"] & b \\ b \ominus a \arrow[ur] \arrow[urr, bend right, "a"'] & & \end{tikzcd}$$

i.e. $b \ominus (a \vee (b \ominus a)) = (b \ominus a) \wedge a$

$(a \vee (b \ominus a)) \oplus ((b \ominus a) \wedge a) = b$ ∎